3.3 The M-Method

Example 3.3 illustrates that the starting basic solution may sometimes be infeasible. The M-method and the Two-phase method discussed in this and the next sections are methods that can find a starting basic feasible solution whenever it exists. Consider again an LPP where there is no desirable starting identity matrix.

$$\begin{array}{ll} \max & x_0 = \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \begin{cases} A\mathbf{x} = \mathbf{b}, \\ \quad \mathbf{x} \geq \mathbf{0}. \end{cases} \end{array}$$

where $\mathbf{b} \geq \mathbf{0}$. We may add suitable number of artificial variables $x_{a_1}, x_{a_2}, \cdots, x_{a_m}$ to it to get a starting identity matrix. The corresponding prices for the artificial variables are $-M$ for maximization problem, where M is sufficiently large. The effect of the constant M is to penalize any artificial variables that will occur with positive values in the final optimal solutions. Using the idea, the LPP becomes

$$\begin{array}{ll} \max & z = \mathbf{c}^T\mathbf{x} - M \cdot \mathbf{1}^T\mathbf{x}_a \\ \text{subject to} & \begin{cases} A\mathbf{x} + I_m\mathbf{x}_a = \mathbf{b} \\ \qquad\qquad \mathbf{x} \geq \mathbf{0}, \end{cases} \end{array}$$

where $\mathbf{x}_a = (x_{a_1}, x_{a_2}, \cdots, x_{a_m})^T$ and $\mathbf{1}$ is the vector of all ones. We observe that $\mathbf{x} = \mathbf{0}$ and $\mathbf{x}_a = \mathbf{b}$ is a feasible starting BFS. Moreover, any solution to $A\mathbf{x} + I_m\mathbf{x}_a = \mathbf{b}$ which is also a solution to $A\mathbf{x} = \mathbf{b}$ must have $\mathbf{x}_a = \mathbf{0}$. Thus, we have to drive $\mathbf{x}_a = \mathbf{0}$ if possible.

Example 3.4. Consider the LP in Example 3.3 again.

$$\begin{array}{ll} \max & x_0 = x_1 + x_2 - 0x_3 - Mx_4 - Mx_5 \\ \text{subject to} & \begin{cases} 2x_1 + x_2 \geq 4 \\ x_1 + 2x_2 = 6 \\ x_1, x_2 \geq 0 \quad x_3 \geq 0,\ x_4,\ x_5 \geq 0 \end{cases} \end{array}$$

Introducing surplus variable x_3 and artificial variables x_4 and x_5 yields,

$$\begin{cases} 2x_1 + x_2 - x_3 + x_4 = 4 \\ x_1 + 2x_2 + x_5 = 6 \\ x_0 - x_1 - x_2 + Mx_4 + Mx_5 = 0 \end{cases}$$

Now the columns corresponding to x_4 and x_5 form an identity matrix. In tableau form, we have

	x_1	x_2	x_3	x_4	x_5	b
x_4	2	1	−1	1	0	4
x_5	1	2	0	0	1	6
x_0	−1	−1	0	M	M	0

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 6 \end{pmatrix}$$

This is not a simplex tableau

Notice that in the x_0 row, the reduced cost coefficients that correspond to the basic variables x_4 and x_5 are not zero. These nonzero entries are to be eliminated first before we have our starting tableau. After eliminations of those M, we have the initial tableau:

$x_0 = x_1 + x_2 + 0x_3 - Mx_4 - Mx_5$

$x_0 = (1+3M)x_1 + (1+3M)x_2 - Mx_3 + 0x_4 + 0x_5 - 10M$

artificial variables $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 6 \end{pmatrix}$ $4/2$, $6/1$

	x_1	x_2	x_3	x_4	x_5	**b**
x_4	2^*	1	-1	1	0	4
x_5	1	2	0	0	1	6
x_0	$-(1+3M)$	$-(1+3M)$	M	0	0	$-10M$
	-3	-3	1			-10

This is a simplex tableau

We note that once an artificial variable becomes non-basic, it can be dropped from consideration in subsequent calculations.

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cancel{x_4} \\ x_5 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 4 \end{pmatrix}$ a.v

	x_1	x_2	x_3	x_5	**b**
x_1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	2
x_5	0	$\frac{3}{2}^*$	$\frac{1}{2}$	1	4
x_0	0	$-\frac{1+3M}{2}$	$-\frac{1+M}{2}$	0	$2-4M$
		$-3/2$	$-1/2$	0	-4

$2/\tfrac{1}{2} = 4$

$4/\tfrac{3}{2} = 8/3$

After we eliminate all the artificial variables we have

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cancel{x_5} \end{pmatrix} = \begin{pmatrix} * \\ * \\ 0 \\ 0 \end{pmatrix}$ a.v

	x_1	x_2	x_3	**b**
x_1	1	0	$-\frac{2}{3}$	$\frac{2}{3}$
x_2	0	1	$\frac{1}{3}^*$	$\frac{8}{3}$
x_0	0	0	$-\frac{1}{3}$	$\frac{10}{3}$

BFS of original problem $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 8/3 \\ 0 \end{pmatrix}$

Simplex tableau of the original problem

At this point all artificial variables are dropped from the problem, and $\mathbf{x} = [2/3, 8/3, 0]^T$ is an initial BFS. Notice that this is the same as Tableau 1 in Example 3.3. After one iteration, we get the final optimal tableau.

	x_1	x_2	x_3	**b**
x_1	1	2	0	6
x_3	0	3	1	8
x_0	0	1	0	6

$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 8 \end{pmatrix}$

Thus the optimal solution is $\mathbf{x}^* = (6, 0, 8)^T$ with $x_0^* = 6$.

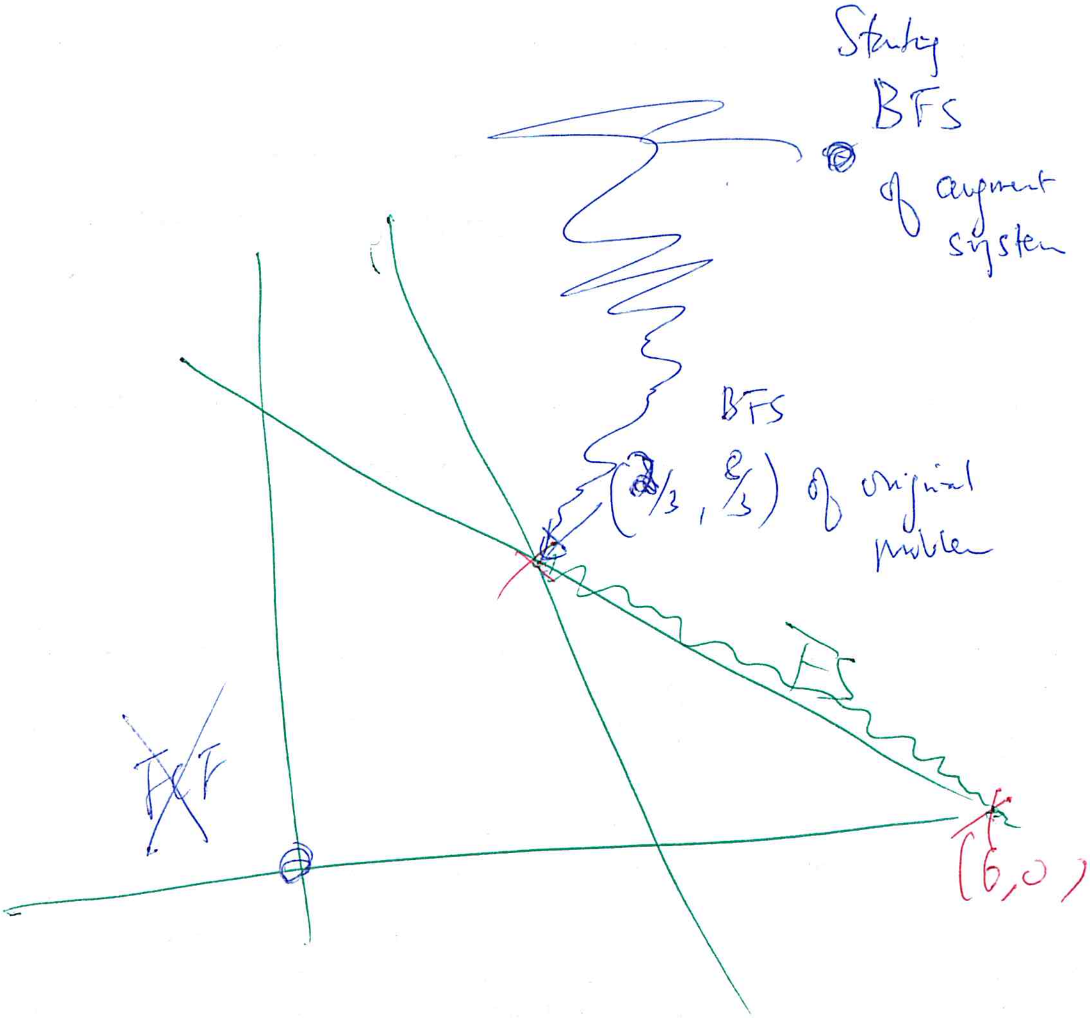
3.4 The Two-Phase Method

The M-method is sensitive to round-off error when being implemented on computers. The two-phase method is used to circumvent this difficulty.

PHASE I: (Search for a Starting BFS)

Instead of considering the actual objective function in the M-Method

$$z = \sum_{i=1}^{n} c_i x_i - M \sum_{i=1}^{m} x_{a_i} ,$$



Eg 3.4

Max $x_0 = x_1 + x_2 + 0x_3 - Mx_4 - Mx_5$

Max $x_0 = \cancel{M}\left(\frac{1}{M}x_1 + \frac{1}{M}x_2 + \frac{0}{M}x_3 - x_4 - x_5\right)$

Phase I

artificial variable

$$\left.\begin{array}{l} \text{Max } x_0 = -x_4 - x_5 \\ \begin{cases} 2x_1 + x_2 - x_3 + x_4 = 5 \\ x_1 + 2x_2 + x_5 = 6 \end{cases} \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{array}\right\} \rightarrow \text{Solve for } x_4 = x_5 = 0$$

	1 0 0 1	
	+1 +1	

→

Optimal ⇓ Solution for Phase I

⟺ a BFS Solution for original problem